Fully-Diverse $Sp(2)$ Code Design

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Abstract — **A fully-diverse code that is suitable for differential modulation for four-transmit-antenna communication systems is constructed based on the symplectic group $Sp(2)$. The code can be regarded as an extension of Alamouti's celebrated two-transmit-antenna orthogonal design which can be constructed from the group $Sp(1)$. The structure of the code lends itself to efficient ML decoding via the sphere decoding algorithm.**

I. Introduction

Consider a wireless communication system with M transmit antennas and N receive antennas. A scheme called differential USTM that works without channel knowledge was proposed in [1], in which the transmitted signal S is a unitary matrix. The design problem is to find a fully-diverse set $\mathcal{C}$ of 2^{MR}, $M \times M$ unitary matrices, where R is the rate of the code. In [2, 3], a group structure was enforced on the constellation which greatly simplifies the problem. However, no good constellations are obtained for very high rates from the finite fpf groups classified in [2] and constellations based on infinite fpf Lie groups are constrained to one and two-transmit-antenna systems[3]. Here, we get high rate constellations for four-transmit-antenna systems, which is a fully-diverse subset of the symplectic group $Sp(2)$. The codes lend themselves to polynomial-time ML decoding via sphere decoder and the simulated performance of the code is compared with existing methods including Alamouti's scheme, Cayley differential unitary space-time codes and group based codes.

II. $Sp(2)$ code design

The nth order symplectic group, $Sp(n)$, is the set of complex $2n \times 2n$ matrices S that is both unitary, $S^*S = SS^* = I_{2n}$, and symplectic, $S^tJS = J$, where $J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}$. $Sp(n)$ has rank n and dimension $n(2n+1)$. Using the two properties, the following theorem corresponding to the parametrization of $Sp(n)$ can be proven.

Theorem 1 (Parametrization of $Sp(n)$). *A matrix S belongs to $Sp(n)$ iff it can be written as*

$$S = \begin{bmatrix} U\Sigma_A V & U\Sigma_B \bar{V} \\ -\bar{U}\Sigma_B V & \bar{U}\Sigma_A \bar{V} \end{bmatrix}$$

where U and V are any $n \times n$ unitary matrices, and $\Sigma_A = \mathrm{diag}(\cos\theta_1, \dots \cos\theta_n)$, $\Sigma_B = \mathrm{diag}(\sin\theta_1, \dots \sin\theta_n)$ for some real angles $\theta_1, \theta_2 \dots, \theta_n$. $\bar{U}$ and $\bar{V}$ denote the conjugates of U and V.

Based on Theorem 1, the matrices in $Sp(n)$ can be parameterized by the entries of U, V and the θ_is. Here, we are mostly interested in the case of $n = 2$. For simplicity, we first let $\Sigma_A = \Sigma_B = \frac{1}{\sqrt{2}}I_2$ and choose U and V as orthogonal designs with P-PSK and shifted Q-PSK entries. The following code, which can be regarded as a generalization of the orthogonal design [1], is obtained.

$$\mathcal{C}_{P,Q,\theta} = \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} UV & U\bar{V} \\ -\bar{U}V & \bar{U}\bar{V} \end{bmatrix} \right\} \tag{1}$$

[1]Matrices in the code have the form $\begin{bmatrix} A & B \\ -\bar{B} & \bar{A} \end{bmatrix}$, but here A and B are 2×2 matrices instead of scalars.

where P and Q are positive integers,

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{j\frac{2\pi k}{P}} & e^{j\frac{2\pi l}{P}} \\ -e^{-j\frac{2\pi l}{P}} & e^{-j\frac{2\pi k}{P}} \end{bmatrix}, V = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{j(\frac{2\pi m}{Q}+\theta)} & e^{j(\frac{2\pi n}{Q}+\theta)} \\ -e^{-j(\frac{2\pi n}{Q}+\theta)} & e^{-j(\frac{2\pi m}{Q}+\theta)} \end{bmatrix} \tag{2}$$

for $0 \le k, l < P, 0 \le m, n < Q, \theta \in [0, 2\pi)$. The rate of the code is $\frac{1}{2}(\log_2 P + \log_2 Q)$. The angle θ is an extra degree of freedom added to the code to obtain the maximal diversity product.

Since the U and V in (2) have an orthogonal design structure, it is not difficult to calculate the determinant of the difference of any two signals in the code directly. Using this calculation, we can prove the following theorem.

Theorem 2 (Condition for full diversity). *There exists a θ such that the code $\mathcal{C}_{P,Q,\theta}$ in (1) is fully-diverse iff P and Q are relatively prime.*

To get codes at higher rates, we can add in one degree of freedom in Σ_A and Σ_B by letting $\Sigma_A = \cos\gamma_i I_2, \Sigma_B = \sin\gamma_i I_2$ for $\gamma_i \in \Gamma$. The full diversity of the modified codes can be proved similarly when θ and the set Γ are properly chosen.

Since the U and V matrices are orthogonal designs, the ML decoder can be reduced to some formula that is quadratic in the real and imaginary parts of entries in U and V. Therefore, the decoding can be done in polynomial time by the sphere decoder.

III. Simulation results

Fig. 1 shows that the $Sp(2)$ code is better than the differential Cayley code, even though the latter has a lower rate, the group diagonal code and the orthogonal design. But it is worse than the $K_{1,1,-1}$ group code[2]. However, decoding the $K_{1,1,-1}$ code requires an exhaustive search over the entire constellation.

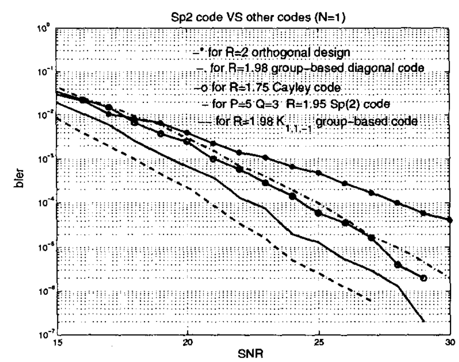

Figure 1: Performance comparison at $R \approx 2$

References

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